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Intuitionistic (T, S)-fuzzy CI-algebras

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ABSTRACT

In this paper, we introduce the notion of intuitionistic (T, S)-fuzzy subalgebras in CI-algebras and study their fundamental properties. We get a fuzzy subalgebra from an intuitionistic (T, S)-fuzzy subalgebra. Also the notion of intuitionistic (T, S)-fuzzy (closed) filters of CI-algebras is introduced. We investigate the relationship between intuitionistic (T, S)-fuzzy subalgebras and intuitionistic (T, S)-fuzzy (closed) filters of CI-algebras.

1. Introduction and preliminaries

Imai and Iseki [1] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. BCI-algebras as a class of logical algebras are the algebraic formulations of the set difference together with its properties in set theory and the implicational functor in logical systems. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

Recently, Kim and Kim defined a BE-algebra [2]. Biao Long Meng, defined the notion of CI-algebra as a generalization of a BE-algebra [3]. In [4], Kim studied on this algebra in detail and some fundamental properties of CI-algebras are discussed, and studied in many papers [5–8].

After the concept of fuzzy sets was introduced by Zadeh [9], several studies were conducted on the generalization of the notion of fuzzy sets. The idea of "intuitionistic fuzzy set" was first introduced by Atanassov [10,11], as a generalization of the notion of fuzzy set. The authors studied some fuzzy algebraic structures [12,13].

Motivated by this, in this paper by using t-norm T and s-norm S, we introduce the notion of intuitionistic (T, S)-fuzzy subalgebras of CI-algebras and intuitionistic (T, S)-fuzzy closed filters of CI-algebras.

Now, we rewrite the basic definitions and some elementary aspects that are necessary for the sequel.

Recall that a CI-algebra is an algebra $(X; *, 1)$ of type $(2, 0)$ satisfying the following axioms:

(CI1) $x * x = 1$;
(CI2) $1 * x = x$;
(CI3) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

In any CI-algebra X one can define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 1$.

A CI-algebra X has the following properties:

(2.1) $y * ((y * x) * x) = 1$,
(2.2) $(x * 1) * (y * 1) = (x * y) * 1$,
(2.3) $1 \leq x \Rightarrow x = 1$ for all $x, y \in X$.

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A non-empty subset S of a CI-algebra X is called a subalgebra of X if $x * y \in S$ whenever $x, y \in S$. A mapping $f : X \to Y$ of a CI-algebra is called a homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

A non-empty subset F of CI-algebra X is called a filter of X if (1) $1 \in F$, (2) $x \in F$ and $x * y \in F$ implies $y \in F$. A filter F of CI-algebra X is said to be closed if $x \in F$ implies $x * 1 \in F$.

Now, we review some fuzzy logic concepts.

A fuzzy set μ in X, i.e., a mapping $\mu : X \to [0, 1]$, and the complement of μ, denoted by $\bar{\mu}$, is the fuzzy set in X by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in X$. For any $\alpha \in [0, 1]$ and a fuzzy set μ in a nonempty set X, the set

$$U(\mu; \alpha) = \{x \in X : \mu(x) \geq \alpha\} \quad (\text{resp. } L(\mu; \alpha) = \{x \in X : \mu(x) \leq \alpha\})$$

is called an upper(resp. lower) level set of μ. Fuzzy set μ is called a fuzzy subalgebra of X if $\mu(x * y) \geq \min(\mu(x), \mu(y))$, for all $x, y \in X$. Note that if μ is a fuzzy subalgebra of a CI-algebra X, then $\mu(1) \geq \mu(x)$, for all $x \in X$.

An intuitionistic fuzzy set (briefly, *IFS*) A in a nonempty set X is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) | x \in X\}$$

where the function $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership and the degree of non-membership, respectively, where

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1$$

for all $x \in X$.

An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) | x \in X\}$ in X can be identified with an ordered pair (μ_A, γ_A) in $I^X \times I^X$. For the sake of simplicity, we shall use symbol $A = \{(x, \mu_A(x), \gamma_A(x)) | x \in X\}$.

Fuzzy set μ is called a fuzzy subalgebra of X with respect to a t-norm T (briefly, a T-fuzzy subalgebra of X) if $\mu(x * y) \geq T(\mu(x), \mu(y))$ for all $x, y \in X$.

Every t-norm T has a useful property: $T(\alpha, \beta) \leq \min(\alpha, \beta)$, for all $\alpha, \beta \in [0, 1]$.

Fuzzy set μ is called a fuzzy subalgebra of X with respect to an s-norm S (briefly, an S-fuzzy subalgebra of X) if $\mu(x * y) \leq S(\mu(x), \mu(y))$ for all $x, y \in X$.

Every s-norm S has a useful property: $S(\alpha, \beta) \geq \max(\alpha, \beta)$, for all $\alpha, \beta \in [0, 1]$.

For a t-norm (or s-norm) P on $[0, 1]$, denote by Δ_P the set of elements $\alpha \in [0, 1]$ such that $P(\alpha, \alpha) = \alpha$, i.e., $\Delta_P := \{\alpha \in [0, 1] | P(\alpha, \alpha) = \alpha\}$.

Definition 1.1 ([14]). Let P be a t-norm (or s-norm). A fuzzy set μ in X is said to satisfy the imaginable property with respect to P if $\text{Im}(\mu) \subseteq \Delta_P$.

2. Intuitionistic (T, S)-fuzzy subalgebras of CI-algebras

In what follows, let X denote a CI-algebra, T be a t-norm and S be an s-norm in $[0, 1]$ unless otherwise specified.

Definition 2.1. Let $A = (\mu_A, \gamma_A)$ be an *IFS* in X. A is called an intuitionistic (T, S)-fuzzy subalgebra of X if

(F1) $\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y))$;
(F2) $\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y))$,

for all $x, y \in X$.

Example 2.2. Let $X := \{1, a, b, c\}$. Define a binary operation "$*$" on X by the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1

Then $(X; *, 0)$ is a CI-algebra. Let $T : [0, 1] \times [0, 1] \to [0, 1]$ be a function defined by

$$T(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$$

and $S : [0, 1] \times [0, 1] \to [0, 1]$ be a function defined by

$$S(\alpha, \beta) = \min(\alpha + \beta, 1)$$

for all $\alpha, \beta \in [0, 1]$. Then T is a t-norm and S is an s-norm. Define an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ by $\mu_A(1) = \mu_A(b) = \mu_A(c) = 1$, $\mu_A(a) = 0$ and $\gamma_A(1) = \gamma_A(b) = \gamma_A(c) = 0$, $\gamma_A(a) = 1$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy subalgebra of X.

Theorem 2.3. *If $\{A_i\}_{i \in I}$ is a family of intuitionistic (T, S)-fuzzy subalgebras of X, then $\bigcap_{i \in I} A_i$ is an intuitionistic (T, S)-fuzzy subalgebra of X, where $\bigcap_{i \in I} A_i = (\vee \mu_i, \wedge \gamma_i)$.*

Proof. Let $x, y \in X$. Then

$$\vee\mu_i(x * y) \geq \vee(T(\mu_i(x), \mu_i(y))) = T(\vee\mu_i(x), \vee\mu_i(y))$$

and

$$\wedge\gamma_i(x * y) \leq \wedge(S(\gamma_i(x), \gamma_i(y))) = S(\wedge\gamma_i(x), \wedge\gamma_i(y)).$$

Hence $\bigcap_{i\in I} A_i = (\vee\mu_i, \wedge\gamma_i)$ is an intuitionistic (T, S)-fuzzy subalgebra of X. □

Proposition 2.4. *Any subalgebra of X can be realized as both a μ_A level subalgebra and γ_A of some intuitionistic (T, S)-fuzzy subalgebra of X.*

Proof. Let A be a subalgebra of X and μ_A, γ_A be fuzzy sets in X defined by

$$\mu_A(x) = \begin{cases} \alpha & \text{if } x \in A; \\ 0 & \text{otherwise} \end{cases}$$

and

$$\gamma_A(x) = \begin{cases} \beta & \text{if } x \in A; \\ 1 & \text{otherwise} \end{cases}$$

for all $x \in X$ where α and β are fixed numbers in $(0, 1)$ such that $\alpha + \beta < 1$.

If $x, y \in A$, then $x * y \in A$. Hence $\mu_A(x) = \mu_A(y) = \mu_A(x * y) = \alpha$ and $\gamma_A(x) = \gamma_A(y) = \gamma_A(x * y) = \beta$.

If at least one of x or y does not belong to A, then at least one of $\mu_A(x)$ or $\mu_A(y)$ is equal to 0 and at least one of $\gamma_A(x)$ or $\gamma_A(y)$ is equal to 1. Therefore $\min(\mu_A(x), \mu_A(y)) = 0$. It follows that $\mu_A(x * y) \geq 0 = \min(\mu_A(x), \mu_A(y))$ and $\gamma_A(x * y) \leq 1 = \max(\gamma_A(x), \gamma_A(y))$. Hence $A = (\mu_A, \gamma_A)$ is an intuitionistic (min, max)-fuzzy subalgebra of X. Obviously, $U(\mu_A, \alpha) = A = L(\gamma_A, \beta)$. □

Theorem 2.5. *If A is a subalgebra of X, then $\bar{I} = (\chi_A, \bar{\chi}_A)$ is an intuitionistic (T, S)-fuzzy subalgebra of X.*

Proof. If $x, y \in A$, then $x * y \in A$. Hence

$$\chi_A(x * y) = 1 \geq T(\chi_A(x), \chi_A(y)).$$

Also, we have

$$0 = 1 - \chi_A(x * y) = \bar{\chi}_A(x * y) \leq S(\bar{\chi}_A(x), \bar{\chi}_A(y)).$$

If $x \in A$ and $y \notin A$, (or, $x \notin A$ and $y \in A$), then $\chi_A(x) = 1$, or $\chi_A(y) = 0$. Thus we have

$$\chi_A(x * y) \geq T(\chi_A(x), \chi_A(y)) = T(1, 0) = 0.$$

Next we have

$$S(\bar{\chi}_A(x), \bar{\chi}_A(y)) = S(1 - \chi_A(x), 1 - \chi_A(y)) = S(0, 1) = 1 \geq \bar{\chi}_A(x * y). \quad \square$$

Theorem 2.6. *Let A be a nonempty subset of X. If $\bar{A} = (\chi_A, \bar{\chi}_A)$ satisfies (F1) or (F2), then A is a subalgebra of X.*

Proof. Suppose that $\bar{A} = (\chi_A, \bar{\chi}_A)$ satisfy (F1) and $x, y \in A$, then it follows from (F1) that

$$\chi_A(x * y) \geq T(\chi_A(x), \chi_A(y)) = T(1, 1) = 1$$

so that $\chi_A(x * y) = 1$, i.e., $x * y \in X$. Hence A is a subalgebra of X. Now suppose that $\bar{A} = (\chi_A, \bar{\chi}_A)$ satisfy (F2). If $x, y \in A$, then by (F2), we have

$$\bar{\chi}_A(x * y) \leq S(\bar{\chi}_A(x), \bar{\chi}_A(y)) \leq S(1 - \chi_A(x), 1 - \chi_A(y)) = S(0, 0) = 0,$$

and thus $\bar{\chi}_A(x * y) = 1 - \chi_A(x * y) = 0$, i.e., $\chi_A(x * y) = 1$. □

Theorem 2.7. *Let A be a fuzzy subalgebra with membership function μ_A in X. Then $\bar{A}$ is an intuitionistic (T, S)-fuzzy subalgebra of X, where $\bar{A} = (\mu_A, \bar{\mu}_A)$.*

Proof. It is sufficient to show that $\bar{\mu}$ satisfies the condition (F2). If $x, y \in X$, then

$$\begin{aligned}\bar{\mu}(x * y) &= 1 - \mu_A(x * y) \leq 1 - T(\mu_A(x), \mu_A(y)) \\ &= S(1 - \mu_A(x), 1 - \mu_A(y)) \\ &= S(\bar{\mu}(x), \bar{\mu}(y)).\end{aligned}$$

Hence $\bar{A}$ is an intuitionistic (T, S)-fuzzy subalgebra of X. □

Theorem 2.8. *An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy subalgebra of X if and only if the fuzzy subsets μ_A and $\bar{\gamma}_A$ are T-fuzzy subalgebras of X.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic (T, S)-fuzzy subalgebra of X. Then clearly μ_A is a T-fuzzy subalgebra of X. Now, for all $x, y \in X$,

$$\begin{aligned}\bar{\gamma}_A(x * y) &= 1 - \gamma_A(x * y) \geq 1 - S(\gamma_A(x), \gamma_A(y))\\ &= T(1 - \gamma_A(x), 1 - \gamma_A(y))\\ &= T(\bar{\gamma}_A(x), \bar{\gamma}_A(y)).\end{aligned}$$

Then $\bar{\gamma}_A$ is a T-fuzzy subalgebra of X.

Conversely, assume that μ_A and $\bar{\gamma}_A$ are T-fuzzy subalgebras of X. It is enough to prove that $\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y))$, for all $x, y \in X$.

Since $\bar{\gamma}_A$ is a T-fuzzy subalgebra of X, then

$$\begin{aligned}\bar{\gamma}_A(x * y) &= 1 - \gamma_A(x * y) \geq 1 - T(\bar{\gamma}_A(x), \bar{\gamma}(y))\\ &= T(1 - \gamma_A(x), 1 - \gamma_A(y))\\ &= 1 - S(\gamma_A(x), \gamma_A(y)).\end{aligned}$$

Hence for all $x, y \in X$,

$$\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y)). \quad \square$$

Definition 2.9. An intuitionistic (T, S)-fuzzy subalgebra $A = (\mu_A, \gamma_A)$ is called an intuitionistic imaginable (T, S)-fuzzy subalgebra of X if μ_A and γ_A satisfy the imaginable property with respect to T and S respectively.

Example 2.10. In Example 2.2, $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of X.

Example 2.11. Let $X := \{1, a, b, c\}$. Define a binary operation "$*$" on X by the following Cayley table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	c	c	c	1

Then $(X, *, 1)$ is a *CI*-algebra. Define an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ by

$$\mu_A(x) = \begin{cases} 0.7 & \text{if } x \in \{1, c\};\\ 0.2 & \text{otherwise}\end{cases}$$

and

$$\gamma_A(x) = \begin{cases} 0.2 & \text{if } x \in \{1, c\};\\ 0.7 & \text{otherwise.}\end{cases}$$

Let $T : [0, 1] \times [0, 1] \to [0, 1]$ be a function defined by

$$T(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$$

for all $\alpha, \beta \in [0, 1]$ and $S : [0, 1] \times [0, 1] \to [0, 1]$ be a function defined by

$$S(\alpha, \beta) = \min(\alpha + \beta, 1)$$

for all $\alpha, \beta \in [0, 1]$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy subalgebra of X which is not imaginable. Because

$$T(\mu_A(a), \mu_A(a)) = T(0.2, 0.2) = \max(0.2 + 0.2 - 1, 0) = 0 \neq \mu_A(a) = 0.2.$$

Proposition 2.12. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of X, then we have $\mu_A(x * 1) \geq \mu_A(x)$ and $\gamma_A(x * 1) \leq \gamma_A(x)$ for all $x \in X$.*

Proof. For any $x \in X$, we have

$$\begin{aligned}\mu_A(x * 1) &\geq T(\mu_A(1), \mu_A(x)) \geq T(\mu_A(x * x), \mu_A(x))\\ &= T(T(\mu_A(x), \mu_A(x)), \mu_A(x)) = \mu_A(x)\end{aligned}$$

and

$$\begin{aligned}\gamma_A(x*1) &\le S(\gamma_A(1), \gamma_A(x)) \le S(\gamma_A(x*x), \gamma_A(x))\\ &= S(S(\gamma_A(x), \gamma_A(x)), \gamma_A(x)) = \gamma_A(x). \quad \square\end{aligned}$$

Proposition 2.13. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of X, then we have $\mu_A(1) \ge \mu_A(x)$ and $\gamma_A(1) \le \gamma_A(x)$, for all $x \in X$.*

Proof. For every $x \in X$, we have

$$\mu_A(1) = \mu_A(x*x) \ge T(\mu_A(x), \mu_A(x)) = \mu_A(x),$$

and

$$\gamma_A(1) = \gamma_A(x*x) \le S(\gamma_A(x), \gamma_A(x)) = \gamma_A(x). \quad \square$$

Proposition 2.14. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of X, then the set*

$$X_A = \{x \in X | \mu_A(x) = \mu_A(1), \gamma_A(x) = \gamma_A(1)\}$$

is a subalgebra of X.

Proof. If $x, y \in X_A$, then $\mu_A(x) = \mu_A(y) = \mu_A(1)$ and $\gamma_A(x) = \gamma_A(y) = \gamma_A(1)$. Since $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of X, it follows that

$$\begin{aligned}&\mu_A(x*y) \ge T(\mu_A(x), \mu_A(y)) = T(\mu_A(1), \mu_A(1)) = \mu_A(1),\\ &\gamma_A(x*y) \le S(\gamma_A(x), \gamma_A(y)) = S(\gamma_A(1), \gamma_A(1)) = \gamma_A(1),\end{aligned}$$

so that $\mu_A(x*y) = \mu_A(1)$ and $\gamma_A(x*y) = \gamma_A(1)$. Thus $x*y \in X_A$, and consequently X_A is a subalgebra of X. $\square$

Theorem 2.15. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic (T, S)-fuzzy subalgebra of X and $\alpha \in [0, 1]$. Then we have*

(i) *if $\alpha = 1$, then the upper level set $U(\mu_A; \alpha)$ is either empty or a subalgebra of X;*
(ii) *if $\alpha = 0$, then the lower level set $L(\gamma_A; \alpha)$ is either empty or a subalgebra of X;*
(iii) *if $T = \min$, then the upper level set $U(\mu_A; \alpha)$ is either empty or a subalgebra of X;*
(iv) *if $S = \max$, then the lower level set $L(\gamma_A; \alpha)$ is either empty or a subalgebra of X.*

Proof. (i) Suppose that $\alpha = 1$ and $x, y \in U(\mu_A; \alpha)$, then $\mu_A(x) \ge \alpha = 1$ and $\mu_A(y) \ge \alpha = 1$. It follows that $\mu_A(x*y) \ge T(\mu_A(x), \mu_A(y)) \ge T(1, 1) = 1$ so that $x*y \in U(\mu_A; \alpha)$. Hence $U(\mu_A; \alpha)$ is a subalgebra of x when $\alpha = 1$.

(ii) Suppose that $\alpha = 0$ and $x, y \in L(\gamma_A; \alpha)$, then $\gamma_A(x) \le \alpha = 0$ and $\gamma_A(y) \le \alpha = 0$. It follows that $\gamma_A(x*y) \le S(\gamma_A(x), \gamma_A(y)) \le S(0, 0) = 0$ so that $x*y \in L(\gamma_A; \alpha)$. Hence $L(\gamma_A; \alpha)$ is a subalgebra of x when $\alpha = 0$.

(iii) Assume that $T = \min$ and $x, y \in U(\mu_A; \alpha)$, then

$$\mu_A(x*y) \ge T(\mu_A(x), \mu_A(y)) = \min(\mu_A(x), \mu_A(y)) \ge \min(\alpha, \alpha) = \alpha$$

for all $\alpha \in [0, 1]$. Hence $x*y \in U(\mu_A; \alpha)$ and so $U(\mu_A; \alpha)$ is a subalgebra of X.

(iv) Let $S = \max$ and $x, y \in L(\gamma_A; \alpha)$. Then

$$\gamma_A(x*y) \le S(\gamma_A(x), \gamma_A(y)) = \max(\gamma_A(x), \gamma_A(y)) \le \max(\alpha, \alpha) = \alpha$$

for all $\alpha \in [0, 1]$. Hence $x*y \in L(\gamma_A; \alpha)$ and so $L(\gamma_A; \alpha)$ is a subalgebra of X.

Let μ be a fuzzy set in X and f be a mapping from X into itself. We define a mapping $\mu^f : X \to [0, 1]$ by $\mu^f(x) = \mu(f(x))$, for all $x \in X$. $\square$

Theorem 2.16. *If $A_a^1 = \{x \in X | \mu_A(x) \ge a\}$ and $A_b^2 = \{x \in X | 1 - \gamma_A(x) \ge b\}$ are subalgebras of X, then A is an intuitionistic (T, S)-fuzzy subalgebra of X.*

Proof. Suppose for any $a \in [0, 1]$, A_a^1 and A_a^2 are subalgebras of X. Let $a := T(\mu_A(x), \mu_A(y))$, for any $x, y \in X$. We have $\mu_A(x), \mu_A(y) \ge a$, then $x, y \in A_a^1$ and A_a^1 is a subalgebra of X then $x*y \in A_a^1$, hence $\mu_A(x*y) \ge a = T(\mu_A(x), \mu_A(y))$.

Also, if $b := T(1 - \gamma_A(x), 1 - \gamma_A(y))$ in a similar way we have

$$\gamma_A(x*y) \le S(\gamma_A(x), \gamma_A(y)). \quad \square$$

In the following example we show that the converse of Theorem 2.16, is not correct in general.

Example 2.17. In Example 2.2, $A_{0.1}^1 = A_{0.5}^2 = \{1, b, c\}$.$A_{0.1}^1 = A_{0.5}^2$ are not subalgebras of X because $b, c \in A_{0.1}^1, A_{0.5}^2$ but $b*c = a \notin A_{0.1}^1, A_{0.5}^2$.

Theorem 2.18. *If A is an intuitionistic $(\min, \max)$-fuzzy subalgebra of X, then both A_a^1, A_b^2 are subalgebras of X.*

Proof. Let $x, y \in A_a^1$. Then $\mu_A(x), \mu_A(y) \geq a$. By hypothesis we have

$$\mu_A(x * y) \geq \min(\mu_A(x), \mu_A(y)) \geq \min(a, a) = a.$$

Hence $x * y \in A_a^1$.
Similarly A_b^2 is a subalgebra of X. $\square$

In the following theorem we show that how we can get a fuzzy subalgebra from an intuitionistic (T, S)-fuzzy subalgebra.

Theorem 2.19. *If A is an intuitionistic* (min, max)*-fuzzy subalgebra of X, then the lower cut set*

$$A_\lambda(x) = \begin{cases} 1 & \text{if } \mu_A(x) \geq \lambda; \\ \dfrac{1}{2} & \text{if } \mu_A(x) < \lambda \leq 1 - \gamma_A(x) \\ 0 & \text{if } \lambda \geq 1 - \gamma_A(x) \end{cases}$$

is a fuzzy subalgebra of X.

Proof. We must show that $A_\lambda(x * y) \geq \min(A_\lambda(x), A_\lambda(y))$. For this, we consider the following cases.

Case (1) If $A_\lambda(x) = A_\lambda(y) = 1$, then $\mu_A(x) \geq \lambda$ and $\mu_A(y) \geq \lambda$. On the other hand, $\mu_A(x * y) \geq \min(\mu_A(x), \mu_A(y)) \geq \lambda$. Therefore $A_\lambda(x * y) = 1 \geq \min(A_\lambda(x), A_\lambda(y))$.

Case (2) If $A_\lambda(x) = 1$ and $A_\lambda(y) = \frac{1}{2}$, then $\mu_A(x) \geq \lambda$ and $\mu_A(y) < \lambda \leq 1 - \gamma_A(x)$. On the other hand, we have $1 - \gamma_A(x * y) \geq \min(1 - \gamma_A(x), 1 - \gamma_A(y))$. By hypothesis we have $1 - \gamma_A(x) \geq \mu_A(x)$, then

$$1 - \gamma_A(x * y) \geq \min(1 - \gamma_A(x), 1 - \gamma_A(y)) \geq \min(\lambda, \lambda) = \lambda.$$

Then $A_\lambda(x * y) = \frac{1}{2} \geq \min(A_\lambda(x), A_\lambda(y))$.

Case (3) If $A_\lambda(x) = A_\lambda(y) = \frac{1}{2}$, then $1 - \gamma_A(x) \geq \lambda$ and $1 - \gamma_A(y) \geq \lambda$. Similarly, we can show $A_\lambda(x*y) \geq \min(A_\lambda(x), A_\lambda(y))$. Therefore A_λ is a fuzzy subalgebra of X. $\square$

Theorem 2.20. *If A is an intuitionistic (T_1, S_1)-fuzzy subalgebra of X, $T_1 \geq T_2$ and $S_1 \leq S_2$, then A is an intuitionistic (T_2, S_2)-fuzzy subalgebra of X.*

Proof. Since A is an intuitionistic (T_1, S_1)-fuzzy subalgebra of X, then

$$\mu_A(x * y) \geq T_1(\mu_A(x), \mu_A(y)) \geq T_2(\mu_A(x), \mu_A(y)).$$

Thus $\mu_A(x * y) \geq T_2(\mu_A(x), \mu_A(y))$.
Similarly $\gamma_A(x * y) \leq S_2(\gamma_A(x), \gamma_A(y))$. $\square$

In the following example we show that the converse of Theorem 2.20, is not correct in general.

Example 2.21. In Example 2.2, an intuitionistic (T, S)-fuzzy subalgebra defined is not an intuitionistic (min, max)-fuzzy subalgebra because

$$\mu_A(b * c) = \mu_A(a) = 0 \ngeq \min(\mu_A(b), \mu_A(c)) = \min(1, 1) = 1.$$

Theorem 2.22. *Let f be an endomorphism of X. If $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of X, then $B = (\mu_A^f, \gamma_A^f)$ is an intuitionistic (T, S)-fuzzy subalgebra of X.*

Proof. For any $x, y \in X$, we have

$$\mu_A^f(x * y) = \mu_A(f(x * y)) = \mu_A(f(x) * f(y)) \geq T(\mu_A(f(x)), \mu_A(f(y))) = T(\mu_A^f(x), \mu_A^f(y)).$$

Similarly, for any $x, y \in X$, we have

$$\gamma_A^f(x * y) = \gamma_A(f(x * y)) = \gamma_A(f(x) * f(y)) \leq S(\gamma_A(f(x)), \gamma_A(f(y))) = S(\gamma_A^f(x), \gamma_A^f(y)). \quad \square$$

Theorem 2.23. *Let $f : X \to Y$ be an epimorphism of CI-algebras. If $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy set in Y. If $B = (\mu_A^f, \gamma_A^f)$ is an intuitionistic (T, S)-fuzzy subalgebra of X, then $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy subalgebra in Y.*

Proof. For any $y_1, y_2 \in Y$, there exist $x_1, x_2 \in X$, such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then

$$\begin{aligned} \mu_A(y_1 * y_2) &= \mu_A(f(x_1) * f(x_2)) = \mu_A(f(x_1 * x_2)) = \mu_A^f(x_1 * x_2) \\ &\geq T(\mu_A^f(x_1), \mu_A^f(x_2)) \\ &= T(\mu_A(f(x_1)), \mu_A(f(x_2))) \\ &= T(\mu_A(y_1), \mu_A(y_2)). \end{aligned}$$

Similarly, we have

$$\begin{aligned}\gamma_A(y_1 * y_2) &= \gamma_A(f(x_1) * f(x_2)) = \gamma_A(f(x_1) * f(x_2)) = \gamma_A^f(x_1 * x_2)\\ &\leq S(\gamma_A^f(x_1), \gamma_A^f(x_2))\\ &= S(\gamma_A(f(x_1)), \gamma_A(f(x_2)))\\ &= S(\gamma_A(y_1), \gamma_A(y_2)). \quad \Box\end{aligned}$$

Theorem 2.24. *Let $A = (\mu_A, \gamma_A)$ be an IFS in X such that the non-empty sets $U(\mu_A; \alpha)$ and $L(\gamma_A; \alpha)$ are subalgebras of X, for all $\alpha \in [0, 1]$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy subalgebra of X.*

Proof. Suppose that there exist $x_0, y_0 \in X$ such that

$$\mu_A(x_0 * y_0) < T(\mu_A(x_0), \mu_A(y_0)).$$

Taking $\alpha_0 := \frac{1}{2}(\mu_A(x_0 * y_0) + T(\mu_A(x_0), \mu_A(y_0)))$, then

$$\min(\mu_A(x_0), \mu_A(y_0)) \geq T(\mu_A(x_0), \mu_A(y_0)) \geq \alpha_0 > \mu_A(x_0 * y_0).$$

It follows that $x_0, y_0 \in U(\mu_A; \alpha_0)$ and $x_0 * y_0 \notin U(\mu_A; \alpha_0)$. This is a contradiction and hence μ_A satisfies the inequality $\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y))$, for all $x, y \in X$.

Similarly, suppose that there exist $x_0, y_0 \in X$ such that

$$\gamma_A(x_0 * y_0) > S(\gamma_A(x_0), \gamma_A(y_0)).$$

Taking $\beta_0 := \frac{1}{2}(\gamma_A(x_0 * y_0) + S(\gamma_A(x_0), \gamma_A(y_0)))$, then

$$\max(\gamma_A(x_0), \gamma_A(y_0)) \leq S(\gamma_A(x_0), \gamma_A(y_0)) \leq \beta_0 < \gamma_A(x_0 * y_0).$$

It follows that $x_0, y_0 \in L(\gamma_A; \beta_0)$ and $x_0 * y_0 \notin L(\gamma_A; \beta_0)$. Hence γ_A satisfies the inequality $\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y))$, for all $x, y \in X$. $\Box$

3. Intuitionistic (T, S)-fuzzy (closed)filters of *CI*-algebras

Definition 3.1. Let $A = (\mu_A, \gamma_A)$ be an *IFS* in X. Then A is called an intuitionistic (T, S)-fuzzy closed filter of *CI*-algebra X if

(F3) $\mu_A(x * 1) \geq \mu_A(x)$ and $\gamma_A(x * 1) \leq \gamma_A(x)$;
(F4) $\mu_A(y) \geq T(\mu_A(x), \mu_A(x * y))$ and $\gamma_A(y) \leq S(\gamma_A(x), \gamma_A(x * y))$,

for all $x, y \in X$.

An intuitionistic (T, S)-fuzzy closed filter $A = (\mu_A, \gamma_A)$ is called an intuitionistic imaginable (T, S)-fuzzy closed filter of X if μ_A and γ_A satisfy the imaginable property with respect to T and S respectively.

Example 3.2. In Example 2.2, $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy closed filter of X.

Proposition 3.3. *Every intuitionistic imaginable (T, S)-fuzzy subalgebra satisfying* (F4) *is an intuitionistic imaginable (T, S)-fuzzy closed filter.*

Theorem 3.4. *Every intuitionistic (T, S)-fuzzy closed filter is an intuitionistic (T, S)-fuzzy subalgebra.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic (T, S)-fuzzy closed filter of X and $x, y \in X$. Then

$$\begin{aligned}\mu_A(x * y) &\geq T(\mu_A(y)), \mu_A(y * (x * y)) = T(\mu_A(y), \mu_A(x * (y * y)))\\ &= T(\mu_A(y), \mu_A(x * 1)) \geq T(\mu_A(y), \mu_A(x))\end{aligned}$$

and

$$\begin{aligned}\gamma_A(x * y) &\leq S(\gamma_A(y)), \gamma_A(y * (x * y)) = S(\gamma_A(y), \gamma_A(x * (y * y)))\\ &= S(\gamma_A(y), \gamma_A(x * 1)) \leq S(\gamma_A(y), \gamma_A(x)).\end{aligned}$$

Hence $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy subalgebra of X. $\Box$

The converse of Theorem 3.4, may not be true.

Example 3.5. Let $X := \{1, a, b, c, d\}$. Define a binary operation "$*$" on X by the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	c	c
c	c	d	1	1	a
d	c	c	c	1	1

Then $(X; *, 0)$ is a CI-algebra. Let $T : [0, 1] \times [0, 1] \to [0, 1]$ be a function defined by

$$T(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$$

for all $\alpha, \beta \in [0, 1]$ and $S : [0, 1] \times [0, 1] \to [0, 1]$ be a function defined by

$$S(\alpha, \beta) = \min(\alpha + \beta, 1)$$

for all $\alpha, \beta \in [0, 1]$. Then T is a t-norm and S is an s-norm. Define an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ by $\mu_A(1) = \mu_A(d) = 0.7$, $\mu_A(a) = \mu_A(b) = \mu_c = 0.07$, and $\gamma_A(a) = \gamma_A(b) = \gamma_A(c) = 0.7$, $\gamma_A(1) = \gamma_A(d) = 0.07$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy subalgebra of X, but it is not an intuitionistic (T, S)-fuzzy closed filter because

$$\mu_A(d * 1) = \mu_A(c) = 0.07 \ngeq \mu_A(d) = 0.7.$$

Theorem 3.6. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic* $(\min, \max)$*-fuzzy closed filter of X. If $x \le y$, then $\mu_A(y) \ge \mu_A(x)$ and $\gamma_A(y) \le \gamma_A(x)$, for any $x, y \in X$.*

Proof. From the assumption, A is an intuitionistic $(\min, \max)$-fuzzy filter of X, so if $x \le y$ then $x * y = 1$. Hence

$$\mu_A(y) \ge \min(\mu_A(x), \mu_A(x * y)) = \min(\mu_A(x), \mu_A(1)) = \mu_A(x).$$

Similarly

$$\gamma_A(y) \le \max(\gamma_A(x), \gamma_A(x * y)) = \max(\gamma_A(x), \gamma_A(1)) = \gamma_A(y). \quad \square$$

Theorem 3.7. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic* $(\min, \max)$*-fuzzy closed filter of X. If $x \le z * y$, then $\mu_A(y) \ge \min(\mu_A(x), \mu_A(z))$ and $\gamma_A(y) \le \max(\gamma_A(x), \gamma_A(z))$, for any $x, y, z \in X$.*

Proof. From the assumption that $\mu_A(y) \ge \min(\mu_A(x), \mu_A(x * y))$, put $y := x * y$ and then we have

$$\mu_A(x * y) \ge \min(\mu_A(z), \mu_A(z * (x * y))),$$

therefore

$$\begin{aligned} \mu_A(y) &\ge \min(\mu_A(x), \mu_A(x * y)) \ge \min(\mu_A(x), \min(\mu_A(z), \mu_A(z * (x * y)))) \\ &= \min(\mu_A(x), \min(\mu_A(z), \mu_A(1))) \\ &= \min(\mu_A(x), \mu_A(z)). \end{aligned}$$

Similarly $\gamma_A(y) \le \max(\gamma_A(x), \gamma_A(x * y))$, put $y := x * y$ and then we have

$$\gamma_A(x * y) \le \max(\gamma_A(z), \gamma_A(z * (x * y))),$$

therefore

$$\begin{aligned} \gamma_A(y) &\le \max(\gamma_A(x), \gamma_A(x * y)) \le \max(\gamma_A(x), \max(\gamma_A(z), \gamma_A(z * (x * y)))) \\ &= \max(\gamma_A(x), \gamma_A(z)). \quad \square \end{aligned}$$

Theorem 3.8. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic (T, S)-fuzzy subalgebra of X. If $A = (\mu_A, \gamma_A)$ satisfies the imaginable property and inequalities $\mu_A(x * y) \le \mu_A(y * x)$ and $\gamma_A(x * y) \ge \gamma_A(y * x)$ for all $x, y \in X$, then $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy closed filter of X.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic (T, S)-fuzzy subalgebra of X which satisfies the inequalities $\mu_A(x*y) \le \mu_A(y*x)$ and $\gamma_A(x * y) \ge \gamma_A(y * x)$ for all $x, y \in X$. It follows from Proposition 2.12, that $\mu_A(x * 1) \ge \mu_A(x)$ and $\gamma_A(x * 1) \le \gamma_A(x)$ for all $x, y \in X$. Then

$$\begin{aligned} \mu_A(y) &= \mu_A(1 * y) \ge \mu_A(y * 1) = \mu_A(y * (x * x)) \\ &= \mu_A(x * (y * x)) \ge T(\mu_A(x), \mu_A(y * x)) \\ &\ge T(\mu_A(x), \mu_A(x * y)), \end{aligned}$$

and

$$\begin{aligned}\gamma_A(y) &= \gamma_A(1*y) \le \gamma_A(y*1) = \gamma_A(y*(x*x)) \\ &= \gamma_A(x*(y*x)) \le S(\gamma_A(x), \gamma_A(y*x)) \\ &\le S(\gamma_A(x), \gamma_A(x*y)).\end{aligned}$$

Hence $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy closed filter of X. □

Theorem 3.9. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic (T, S)-fuzzy subalgebra of X such that the non-empty sets $U(\mu_A; \alpha)$ and $L(\gamma_A; \alpha)$ are closed filters of X and $\alpha \in [0, 1]$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy closed filter of X.*

Proof. Suppose that there exist $x_0, y_0 \in X$ such that

$$\mu_A(y_0) < T(\mu_A(x_0), \mu_A(x_0 * y_0)).$$

Taking $\alpha_0 := \frac{1}{2}(\mu_A(x_0) + T(\mu_A(x_0 * y_0), \mu_A(x_0)))$, then

$$\min(\mu_A(x_0 * y_0), \mu_A(x_0)) \ge T(\mu_A(x_0 * y_0), \mu_A(x_0)) \ge \alpha_0 > \mu_A(y_0).$$

It follows that $x_0 * y_0, x_0 \in U(\mu_A; \alpha_0)$ and $y_0 \notin U(\mu_A; \alpha_0)$. This is a contradiction and hence μ_A satisfies the inequality $\mu_A(x * y) \ge T(\mu_A(x), \mu_A(y))$, for all $x, y \in X$.

Similarly, suppose that there exist $x_0, y_0 \in X$ such that

$$\gamma_A(y_0) > S(\gamma_A(x_0), \gamma_A(x_0 * y_0)).$$

Taking $\beta_0 := \frac{1}{2}(\gamma_A(x_0) + S(\gamma_A(x_0 * y_0), \gamma_A(x_0)))$, then

$$\max(\gamma_A(x_0), \gamma_A(x_0 * y_0)) \le S(\gamma_A(x_0), \gamma_A(x_0 * y_0)) \le \beta_0 < \gamma_A(y_0).$$

It follows that $x_0, x_0 * y_0 \in L(\gamma_A; \beta_0)$ and $y_0 \notin L(\gamma_A; \beta_0)$. This is a contradiction and hence γ_A satisfies the inequality $\gamma_A(x * y) \le S(\gamma_A(x), \gamma_A(y))$ for all $x, y \in X$.

Now assume that there exists $x_0 \in X$ such that $\mu_A(x * 1) < \mu_A(x_0)$. Taking

$$\alpha_0 := \frac{1}{2}(\mu_A(x_0 * 1) + \mu_A(x_0))$$

then $\mu(x_0 * 1) \le \alpha_0$ and $\mu_A(x_0) \ge \alpha_0$. It follows that $x_0 \in U(\mu_A; \alpha_0)$ but $x_0 * 1 \notin U(\mu_A; \alpha_0)$. This is a contradiction. Hence $\mu_A(x * 1) \ge \mu_A(x)$, for all $x \in X$. Similarly, we get that $\gamma_A(x * 1) \le \gamma_A(x)$ for all $x \in X$. □

4. Conclusion

In this paper, we have introduced the concept of intuitionistic (T, S)-fuzzy subalgebras of *CI*-algebras and intuitionistic (T, S)-fuzzy closed filters of *CI*-algebras. Some related properties are investigated, for example a fuzzy subalgebra from an intuitionistic (T, S)-fuzzy subalgebra is constructed. We show that an intuitionistic (T, S)-fuzzy closed filter of a *CI*-algebra is an intuitionistic (T, S)-fuzzy subalgebra but the converse is not true.

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